

CALCULUS II TUTORIAL (9): SOLUTIONS

①

$$f = 2x^2 - 4x + y^2 - 4y + 1$$

$$\Rightarrow f_x = 4x - 4, f_y = 2y - 4 \Rightarrow f_x = f_y = 0 \text{ at } (1, 2)$$

$$\text{Also } f_{xx} = 4, f_{yy} = 2, f_{xy} = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 8$$

Hence there is minimum at $(1, 2)$

$$f = 2 - 4 + 4 - 8 + 1 = -5$$

Also $f = 1$ at $(0, 0)$.

$$\text{On OA: } x = 0 \Rightarrow f = y^2 - 4y + 1 \Rightarrow \frac{df}{dy} = 2y - 4 = 0 \text{ at } y = 2$$

$$\text{Since } \frac{d^2f}{dy^2} = 2, \text{ there is minimum at } (0, 2) \text{ of } f = 4 - 8 + 1 = -3$$

$$\text{On AB: } y = 2 \Rightarrow f = 2x^2 - 4x - 3 \Rightarrow \frac{df}{dx} = 4x - 4 = 0 \text{ at } x = 1$$

$$\text{Since } \frac{d^2f}{dx^2} = 4, \text{ there is minimum at } (1, 2) \text{ of } f = -5$$

$$\text{On OB: } y = 2x \Rightarrow f = 2x^2 - 4x + 4x^2 - 8x + 1 = 6x^2 - 12x + 1$$

$$\Rightarrow \frac{df}{dx} = 12x - 12 = 0 \text{ at } x = 1 \Rightarrow \text{minimum at } (1, 2) \text{ of } f = -5$$

② Absolute minimum is -5 . Absolute maximum is 1 .

$$(a) f = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$$

$$\Rightarrow f_x = 2y - 10x + 4, f_y = 2x - 4y + 4$$

$$f_x = f_y = 0 \Rightarrow \begin{cases} -5x + y = 2 \\ x - 2y = -2 \end{cases} \Rightarrow x = \frac{2}{3}, y = \frac{4}{3}$$

$$f_{xx} = -10, f_{yy} = -4, f_{xy} = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 40 - 4 = 36$$

Hence maximum at $(\frac{2}{3}, \frac{4}{3})$

$$\text{with } f = \frac{16}{9} - \frac{20}{9} - \frac{32}{9} + \frac{8}{3} + \frac{16}{3} - 4 = 0$$

$$(b) f = 9x^3 + \frac{1}{3}y^3 - 4xy \Rightarrow f_x = 27x^2 - 4y, f_y = y^2 - 4x$$

$$f_x = f_y = 0 \Rightarrow \begin{cases} 27x^2 = 4y \\ y^2 = 4x \end{cases} \Rightarrow \frac{27}{16}y^4 = 4y \Rightarrow y = 0 \text{ or } \frac{4}{3}$$

$$f_{xx} = 54x, f_{yy} = 2y, f_{xy} = -4 \Rightarrow \begin{cases} \Rightarrow x = 0 \text{ or } \frac{4}{9} \end{cases}$$

$$\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 108xy - 16$$

Hence saddle at $(0, 0)$ with $f = -4$

$$\text{Minimum at } (\frac{4}{9}, \frac{4}{3}) \text{ with } f = \frac{64}{81} + \frac{64}{81} - \frac{64}{27} = -\frac{64}{81}$$

③ $f = x^2 + 2y + 2z - 4 \Rightarrow \nabla f = (2x, 2, 2) = (2, 2, 2)$ at $(1, 1, \frac{1}{2})$
 $g = y - 1 \Rightarrow \nabla g = (0, 1, 0)$. Hence $\nabla f \wedge \nabla g = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{vmatrix} = (-2, 0, 2)$
Hence tangent line is $x = 1 - 2t, y = 1, z = \frac{1}{2} + 2t$.

④ $f = \ln \sqrt{x^2 + y^2 + z^2} \Rightarrow \nabla f = \frac{(x, y, z)}{x^2 + y^2 + z^2} = \frac{(3, 4, 12)}{9 + 16 + 144} = (\frac{3}{169}, \frac{4}{169}, \frac{12}{169})$
 $\underline{A} = (3, 6, -2) \Rightarrow \hat{\underline{A}} = \frac{\underline{A}}{|\underline{A}|} = (\frac{3}{7}, \frac{6}{7}, -\frac{2}{7}) \Rightarrow \nabla f \cdot \hat{\underline{A}} = \frac{3 \cdot 3 + 4 \cdot 6 - 2 \cdot 12}{7 \cdot 169} = \frac{9}{1183}$
Hence $df = (\nabla f \cdot \hat{\underline{A}}) ds = \frac{9}{1183} \times 0.1 \approx 0.0008$.

⑤ $f = e^x \cos y \Rightarrow f_x = e^x \cos y, f_y = -e^x \sin y$. Linearization is $L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$
 $\Rightarrow L(x, y) = 1 + 1(x - 0) + 0(y - 0) = x + 1$ at $(0, 0)$
 $|f_{xx}| = |e^x \cos y| < 1.1, |f_{xy}| = |e^x \sin y| < 1.1 \times 0.1 = 0.111$
 $|f_{yy}| = |e^x \cos y| < 1.1 \Rightarrow \text{Error} < \frac{1}{2} (1.1) ((0.1) + (0.1))^2 = 0.0222$
 $\downarrow$ maximum 2nd derivative

⑥ $f = xz - 3yz + 2 \Rightarrow \nabla f = (z, -3z, x - 3y) = (2, -6, -2)$ at $(1, 1, 2)$
 $\Rightarrow L(x, y) = (2 - 6 + 2) + 2(x - 1) - 6(y - 1) - 2(z - 2) = 2x - 6y$
 $|f_{xz}| = 1, |f_{yz}| = 3, |f_{zx}| = 1, |f_{zy}| = -3$. $\underline{A} = 2x - 6y - 2z + 6$
Hence error $< \frac{1}{2} \cdot 3 \cdot ((0.01) + (0.01) + (0.02))^2 = 0.0024$.

⑦ $f = 49 - x^2 - y^2$ and $x + 3y = 10$. Eliminating x gives
 $f = 49 - (10 - 3y)^2 - y^2 = -51 + 60y - 10y^2$
 $\Rightarrow f_y = 60 - 20y = 0$ at $y = 3 \Rightarrow f = 39$ (max since $f_{yy} < 0$)
Alternatively put $g = x + 3y - 10$ and solve $\nabla f = \lambda \nabla g$
 $f_x = \lambda g_x \Rightarrow -2x = \lambda, f_y = \lambda g_y \Rightarrow -2y = 3\lambda$
Then $x + 3y = -\frac{\lambda}{2} - \frac{9\lambda}{2} = 10 \Rightarrow \lambda = -2 \Rightarrow x = 1, y = 3$ etc.

⑧ $f = x^2 + y^2$ and $g = x^2 - 2x + y^2 - 4y = 0$. Solve $\nabla f = \lambda \nabla g$.
 $f_x = \lambda g_x \Rightarrow 2x = \lambda(2x - 2) \Rightarrow x = \frac{\lambda}{\lambda - 1}$
 $f_y = \lambda g_y \Rightarrow 2y = \lambda(2y - 4) \Rightarrow y = \frac{2\lambda}{\lambda - 1}$
 $g = 0 \Rightarrow \frac{\lambda^2}{(\lambda - 1)^2} - \frac{2\lambda}{\lambda - 1} + \frac{4\lambda^2}{(\lambda - 1)^2} - \frac{8\lambda}{\lambda - 1} = 0 \Rightarrow \frac{5\lambda^2}{(\lambda - 1)^2} = \frac{10\lambda}{\lambda - 1}$
 $\Rightarrow \lambda = 0 \text{ or } 2$
 $\lambda = 0 \Rightarrow (0, 0) \Rightarrow f = 0$ (min). $\lambda = 2 \Rightarrow (2, 4) \Rightarrow f = 20$ (max)